Int. J. Solids Structures, 1977, Vol. 13, pp. 605–614. Pergamon Press. Printed in Great Britain

ON SINGLE AND BIMODAL OPTIMUM BUCKLING LOADS OF CLAMPED COLUMNS

NIELS OLHOFF and STEEN HØJLUND RASMUSSEN
Department of Solid Mechanics, The Technical University of Denmark, Lyngby, Denmark

(*Received* 2 *September* 1976; *revised* 15 *November* 1976)

Abstract—We study the problem of determining the optimum shape of a thin, elastic, clamped column of given length and volume, such that the fundamental buckling load is a maximum. The column cross-sections are assumed to be geometrically similar, and a minimum allowable value is specified for the cross-sectional area.

Investigating the optimization problem parametrically in terms of this minimum constraint, we reveal a significant feature. There exists a threshold value for the constraint, beyond which the optimum columns are all associated with single mode optimum buckling loads, whereas, for any value of the constraint less than the threshold value, the optimum columns are associated with bimodal fundamental buckling loads.

This bimodal behaviour necessitates an extension and a mathematical reformulation of the current optimization problem, which is outlined and solved in the paper. In particular, we revise the result hitherto considered to be the optimum solution for an unconstrained column with clamped ends.

INTRODUCTION

This paper reconsiders the problem of finding the shape of the strongest elastic column with clamped ends, i.e. the column that has the largest fundamental buckling load from among all similar columns of equal length, volume and material. The problem was dealt with by Tadjbakhsh and Keller[1] in 1962, together with the strongest column problems associated with clamped-free and clamped-simply supported end conditions. The strongest column with simply supported ends was determined in the preceding paper[2] by Keller.

The present study demonstrates that the clamped-clamped case is more complex than hitherto assumed, and that it significantly differs in mathematical formulation from the cases associated with the other sets of classical boundary conditions. Thus, in order to determine the optimum column with clamped ends, it is necessary to take into account the possibility that the optimum fundamental buckling load is a double eigenvalue. This implies that the differential equations governing optimality become bimodal.

In Section 1, we consider the usual single mode formulation of the non-dimensionalized column optimization problem. We first show that the clamped-clamped column arrived at in Ref. [1] is not the optimum solution because the column possesses a buckling mode that becomes critical at a *lower* value of the axial load than the mode considered in the analytical solution of the optimality equations in Ref. [1]. Then, by taking into account a less restrictive class of deflection functions than in Ref. [1], we determine the solution of the single mode optimality equations, which, in the first instance, replaces the solution in Ref. [1]. However, we find that the column hereby determined is still not the optimum solution for the simple reason that there exist clamped columns of other shapes but of equal volume, length and material, which have higher values of the fundamental buckling load.

This leads us, in Section 2, to reformulate and extend the problem under study, including the possibility of the optimum fundamental buckling load being a double eigenvalue, and also taking into account a prescribed minimum allowable value for the column cross-sectional area. This dimensionless minimum area constraint may be assigned any value in the range from 0 to 1, which correspond to unconstrained and fully constrained (uniform) designs, respectively.

Our analytical development is based on a functional formed as the bimodal potential energy of the column in its buckled configuration and terms that represent the volume and the minimum area constraints, respectively. In addition, suitable normalizations of the two modes are included. The bimodal optimization problem is represented through this functional, and the governing equations for the problem are produced by variational analysis. Geometrically unconstrained optimization and/or single mode optimization are contained as special cases of the expanded formulation.

Finally, the results of our analysis are outlined and discussed in Section 3.

1. BASIC CONCEPTS AND RESULTS OBTAINED BY A SINGLE MODE FORMULATION

We consider a thin, straight, elastic column of variable, but geometrically similar (and similarly oriented) cross-sections where the area A and the area moment of inertia I are correlated by $I = cA^2$, with the constant c given by the cross-sectional geometry. The column has the volume V, length L and Young's modulus E, and it is subjected to an axial compressive force, the value of which is P at buckling.

Introducing a dimensionless cross-sectional area $\alpha(x)$ and buckling load λ by

$$\alpha(x) = A(x)L/V \tag{1}$$

$$\lambda = \frac{PL^4}{EcV^2}, \tag{2}$$

where the coordinate x is non-dimensionalized by means of L, the lateral deflection $y(x)$ at buckling is governed by the eigenvalue problem

$$(\alpha^2 y'')'' = -\lambda y'' \tag{3}$$

$$y(0) = y'(0) = y(1) = y'(1) = 0, \tag{4}$$

where the column ends $x = 0$ and $x = 1$ are assumed to be clamped.

A convenient expression for λ, actually the Rayleigh quotient,

$$\lambda = \frac{\int_0^1 \alpha^2 y''^2 \, dx}{\int_0^1 y'^2 \, dx} \tag{5}$$

is obtained if we multiply the differential equation (3) by $y(x)$ and invoke the boundary conditions (4) through two integrations by parts. The property of λ being stationary at the actual deflection $y(x)$ among all other kinematically admissible deflection functions, is, of course, well known. To be kinematically admissible, a deflection $y(x)$ must be continuous, satisfy the kinematic boundary conditions, and have continuous slope y' except possibly at sections of vanishing bending stiffness.

The optimization problem consists in determining the column shape that maximizes the fundamental buckling load P for given values of V, L, E and c. In dimensionless terms, this problem is equivalent to determining the design variable $\alpha(x)$ such that the eigenvalue λ given by (5) is maximized subject to the constraint

$$\int_0^1 \alpha \, dx = 1, \tag{6}$$

see (1), (2).

Forming a functional based on (5), adjoining (6) by means of a Lagrangian multiplier 2β, the condition of stationarity with respect to arbitrary admissible variation of $\alpha(x)$ yields the optimality condition

$$\alpha(x) = \frac{\beta}{y''^2}, \tag{7}$$

and the stationarity with respect to variation of $y(x)$ recovers the differential eqn (3) along with conditions of continuity throughout of the functions $(\alpha^2 y'')' + \lambda y'$ and $\alpha^2 y''$, respectively. At an interior point of vanishing bending moment $m(x) = \alpha^2 y''$, a discontinuity of the slope y' is possible because the cross-section α vanishes (with y'' tending to infinity, see (7)). Clearly, such behaviour of y' would then imply a discontinuity of the shear force $q(x) = -(\alpha^2 y'')'$.

Now, the four coupled differential and integral eqns (3), (5)–(7) and the boundary conditions (4) for the four unknowns $\alpha(x)$, $y(x)$, λ and β, constitute the mathematical formulation of our optimization problem in dimensionless form. *This is the single mode formulation of the optimization problem.*

A precisely equivalent set of governing equations for the clamped-clamped column optimization problem was considered among the cases of clamped-free and clamped-simply supported end conditions in Ref. [1]. The solution obtained in Ref. [1] is illustrated in Fig. 1 by the deflection function $y(x)$, and the linear dimensions of the corresponding column are indicated by the square root of the cross-sectional area function. The value of λ was found to be $\lambda = 16\pi^2/3$ (=52.638), which exceeds by one third the dimensionless buckling load for a uniform column.

The column is symmetrical, and its cross-section $\alpha(x)$ vanishes along with the bending moment $m(x) = \alpha^2 y''$ at the points $x = \frac{1}{4}$ and $x = \frac{3}{4}$, where y'' exhibits singularities. This behaviour is consistent with the governing eqns (3)–(7), and the points of vanishing α and m may be considered inner hinges. The middle part between the hinges has the shape of an optimum column with simply supported ends, viz. Ref. [2] and special cases in Frauenthal[3] and Rasmussen[4]. The two outer parts are shaped as optimum clamped-free columns (Ref. [1]) each of which is identical to half the middle part.

The solution shown in Fig. 1 has been considered the optimum clamped column since its publication in 1962, but, in fact, it is not. Thus, the deflection function $y(x)$ does not correspond to the *fundamental* buckling load of the resulting column. By implicitly assuming *continuity* of the slope $y'(x)$, and hence of the shear force $q(x)$ (see the discussion above), a too narrow class of functions was considered in the analytical solution procedure used in Ref. [1]. As is in complete agreement with the governing equations, especially eqns (3) and (5), jumps of y' and $q(x) = -(\alpha^2 y'')'$ are possible at the points of vanishing bending moment $m(x) = \alpha^2 y''$, and consequently have to be allowed for.

A simple check of the non-optimality of the column in Fig. 1 is provided if we take the column area function $\alpha(x)$ to be given and solve the usual eigenvalue problem, eqns (3) and (4), for the associated fundamental buckling load λ_1 and mode $y_1(x)$. Hereby, we find $\lambda_1 = 30.51$, which is even *lower* than the buckling load $\lambda_1{}^u = 4\pi^2$ (=39.478) of a corresponding *uniform* column. The buckling mode y_1 is antisymmetrical, see Fig. 1, and the middle part of the column only undergoes a rigid body rotation, i.e. it remains undeformed. Thus, the *overall* structural buckling load λ_1 is too small to cause *local* buckling of the middle part. This further indicates non-optimality, since some material might be saved from the middle part without decreasing λ_1.

In order to determine the column satisfying the single mode optimality eqns (3)–(7) together with its *fundamental* mode, we now perform a numerical solution, allowing for possible jumps in the slope and in the shear force at two interior points of zero bending moment. The method is based on a formal integration, followed by a finite difference formulation, that takes into account the singular behaviour of y'' at the inner hinges, and a procedure of successive iterations is used.

The solution is shown in Fig. 2, where the mode y_1 is suitably normalized. The points of vanishing cross-section and bending moment are now found to be placed at $x = 0.208$ and

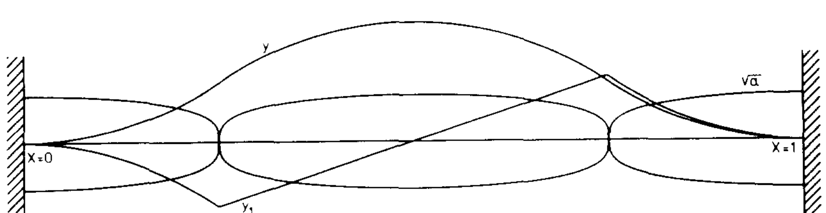

Fig. 1. Reproduction of column shape $\pm\sqrt{\alpha}$ and mode y obtained in Ref. [1]. The true fundamental mode of the column is indicated by y_1. The eigenvalues associated with the modes y_1 and y are 30.51 and 52.638, respectively.

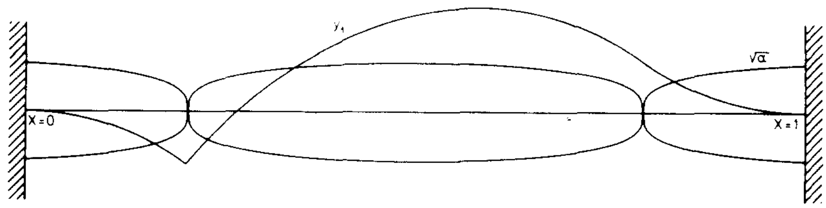

Fig. 2. The solution of the geometrically unconstrained single mode optimization problem. $\lambda_1 = 42.91$ and $\lambda_1/\lambda_1^u = 1.087$.

$x = 0.792$, respectively, and the entire column is symmetrical† within computational accuracy. The middle part includes 61.6% of the total volume, and is still shaped as an optimum, simply supported column, whereas the shapes of the other column parts now deviate slightly from the optimum shape of a clamped-free column. The buckling load of the structure is found to be $\lambda_1 = 42.91$, which is 8.7% higher than the buckling load of a corresponding uniform column.

However, it turns out that the column thus obtained is still not the geometrically unconstrained optimum solution for the problem under study, because there exist differently shaped clamped columns which have higher values of the dimensionless fundamental buckling load λ. This indicates that the single mode formulation is inadequate for the geometrically unconstrained optimization problem. Therefore, we now seek the optimum solution under the assumption that optimum λ is a double eigenvalue, which leads to a *bimodal formulation* of the problem.

2. THE BIMODAL FORMULATION

In this section, we reformulate and expand the mathematical formulation of our optimization problem. We consider the possibility that the dimensionless fundamental buckling load λ, see eqn (2), of the optimum column is a *double* eigenvalue, i.e. that

$$\lambda = \int_0^1 \alpha^2 y_i''^2 \, dx \qquad i = 1, 2, \tag{8}$$

where $y_1(x)$ and $y_2(x)$ denote the two buckling modes corresponding to the optimum λ. When λ is double, y_1 and y_2 need not be mutually orthogonal, but the forms of eqns (8) presuppose the following normalizations of the modes (see eqn 5),

$$\int_0^1 y_i'^2 \, dx = 1 \qquad i = 1, 2. \tag{9}$$

The fixed volume condition for the column is expressed by

$$\int_0^1 \alpha \, dx = 1. \tag{10}$$

Now, aiming at further insight, we will adopt an additional constraint for the design variable $\alpha(x)$, namely that $\alpha(x) \geq \bar{\alpha}$ throughout, assuming the minimum allowable value $\bar{\alpha}$ $(0 \leq \bar{\alpha} \leq 1)$ to be given. This minimum constraint can be expressed by means of the real slack variable $g(x)$ in the standard fashion

$$g^2(x) = \alpha(x) - \bar{\alpha}. \tag{11}$$

We apply a variational formulation of our optimization problem and consider the functional

$$\lambda^* = \int_0^1 \alpha^2 y_1''^2 \, dx - \gamma\left\{\int_0^1 \alpha^2 y_1''^2 \, dx - \int_0^1 \alpha^2 y_2''^2 \, dx\right\} - \sum_{i=1}^{2} \eta_i \left\{\int_0^1 y_i'^2 \, dx - 1\right\} \tag{12}$$

$$- 2\beta\left\{\int_0^1 \alpha \, dx - 1\right\} - 2\int_0^1 \mu(x)\{g^2 - \alpha + \bar{\alpha}\} \, dx,$$

†This was not assumed in the solution procedure.

appending to the Rayleigh quotient based upon y_1, the assumption of the eigenvalue being double (see eqns 8), and the constraints (9), (10) and (11) by means of Lagrangian multipliers γ, η_i $(i = 1, 2)$, 2β and $2\mu(x)$, respectively. The governing equations of the optimization problem are now obtained as the Euler-Lagrange equations following from the stationarity of the functional λ^* with respect to variation of $\alpha(x)$, $g(x)$, $y_1(x)$ and $y_2(x)$, respectively.

Variation of $\alpha(x)$ and $g(x)$ leads to

$$\alpha\{(1-\gamma)y_1''^2 + \gamma y_2''^2\} - \beta + \mu(x) = 0 \tag{13}$$

and

$$\mu(x)g(x) = 0, \tag{14}$$

respectively. Equation (13) is the bimodal optimality condition, which may give us the optimum cross-sectional area function $\alpha(x)$. Before then, however, we remove explicit appearance of the functions $\mu(x)$ and $g(x)$.

Let us identify by x_u and x_c the sub-interval(s) in which we have $g(x) \neq 0$ and $g(x) = 0$, respectively, noting that x_u and x_c make up the entire interval $0 \leq x \leq 1$.

For $x \in x_u$, we then have $\alpha(x) > \bar{\alpha}$, i.e. the cross-sectional area is *unconstrained*, and, moreover, we have $\mu(x) \equiv 0$, which reduces eqn (13). These results follow directly from eqns (11) and (14). For $x \in x_c$, we have $\alpha(x) \equiv \bar{\alpha}$, i.e. the cross-sectional area is *constrained*. Consequently, eqns (13) and (14) may be replaced by

$$\alpha(x) = \begin{cases} \dfrac{\beta}{(1-\gamma)y_1''^2 + \gamma y_2''^2} & (\text{if} > \bar{\alpha}) \quad x \in x_u \\ \bar{\alpha} & \quad x \in x_c. \end{cases} \tag{15}$$

The stationarity of λ^* for arbitrary admissible variations of y_i, $i = 1, 2$, yields the buckling differential equations for y_i in their well known form after identifying the Lagrangian multipliers η_i by means of (8). Along with the clamped end boundary conditions, we thus have

$$\left.\begin{array}{c} (\alpha^2 y_i'')'' = -\lambda y_i'' \\ y_i(0) = y_i'(0) = y_i(1) = y_i'(1) = 0 \end{array}\right\} \quad i = 1, 2 \tag{16}$$

with the functions y_1 and y_2 being subject to the normalizations

$$\int_0^1 y_i'^2 \, dx = 1 \qquad i = 1, 2. \tag{17}$$

Convenient expressions for the Lagrangian multipliers β and γ and the optimum double buckling eigenvalue λ are to be determined next. First, we substitute (15) into the volume constraint (10), thereby obtaining an explicit expression for β,

$$\beta = \frac{1 - \bar{\alpha}\displaystyle\int_{x_c} dx}{\displaystyle\int_{x_u} \frac{dx}{(1-\gamma)y_1''^2 + \gamma y_2''^2}}. \tag{18}$$

Then, subtracting the two equations comprised in (8), substituting $\alpha(x)$ from (15) and using (18), we find the following *implicit* equation for γ,

$$\int_{x_u} \frac{y_1''^2 - y_2''^2}{(1-\gamma)y_1''^2 + \gamma y_2''^2} dx + \bar{\alpha}^2 \left\{ \frac{\displaystyle\int_{x_u} \frac{dx}{(1-\gamma)y_1''^2 + \gamma y_2''^2}}{1 - \bar{\alpha}\displaystyle\int_{x_c} dx} \right\}^2 \int_{x_c} (y_1''^2 - y_2''^2) \, dx = 0. \tag{19}$$

Finally, substitution of (15) and (18) into (8) yields an explicit expression for λ,

$$\lambda = \left\{ \frac{1 - \bar{\alpha} \int_{x_c} \mathrm{d}x}{\int_{x_u} \frac{\mathrm{d}x}{(1-\gamma)y_1''^2 + \gamma y_2''^2}} \right\}^2 \int_{x_u} \frac{y_1''^2}{\{(1-\gamma)y_1''^2 + \gamma y_2''^2\}^2} \mathrm{d}x + \bar{\alpha}^2 \int_{x_c} y_1''^2 \, \mathrm{d}x. \tag{20}$$

Equations (15)–(20) comprise the complete set of necessary equations governing the bimodal optimization problem, and they constitute a strongly coupled, non-linear integro-differential eigenvalue problem. The unknowns to be determined are the optimum double buckling eigenvalue λ, the optimum column cross-sectional area function $\alpha(x)$ (and thereby the sub-intervals x_u and x_c), the eigenfunctions y_1 and y_2, and the Lagrangian multipliers β and γ, respectively. The solution depends in general on the minimum constraint $\bar{\alpha}$, which is the only specified quantity in the non-dimensional formulation.

The method of solution is outlined in the Appendix, and here, we shall briefly discuss some special cases of the extended problem formulated above.

Note first that we may drop the geometric minimum constraint from the formulation by setting $\mu(x) \equiv 0$ in (12), and that the governing equations for the resulting geometrically *unconstrained* optimization problem are then obtained as the special case of eqns (15)–(20) associated with $\bar{\alpha} = 0$ (implying vanishing of the sub-interval(s) x_c).

We may also reduce the bimodal optimization formulation to the *single mode* formulation by specifying $\gamma = 0$ and $\eta_2 = 0$ in the functional λ^*, eqn (12). This leads to a subset of the governing eqns (15)–(20), where eqns (16) and (17) only have to be considered for $i = 1$, and where eqn (19) drops out. Since $\gamma = 0$, y_2 and its derivatives vanish from all equations. Note that the equations governing the geometrically unconstrained single mode optimum buckling load optimization problem, thus obtainable as a special case, were exposed in the previous Section.

Being based on the functional λ^*, eqn (12), the bimodal formulation (15)–(20) possesses a property that is particularly valuable in the present context; it has the ability of handling automatically a problem where the optimum buckling load is, de facto, a simple eigenvalue. Clearly, such behaviour manifests itself by the functions y_1 and y_2 becoming identical, thereby leaving the Lagrangian multiplier γ undetermined (see eqn 12). Note that no condition of, say, linear independence or mutual orthogonality, is imposed upon y_1 and y_2.

Hence, *our general formulation, eqns* (15)–(20), *by solution, directly provides the answer to our* a priori *question, namely, is the optimum column subject to a given value of $\bar{\alpha}$ associated with a simple or a double fundamental buckling load λ.*

3. RESULTS AND DISCUSSION

Subject to various values of the geometric minimum constraint $\bar{\alpha}$, we now solve the governing eqns (15)–(20) of our optimization problem by the numerical method outlined in the Appendix. For any value of $\bar{\alpha}$, we find the optimum column to be *symmetrical* within computational accuracy (symmetry was not assumed in the solution procedure). In Fig. 3, optimum column *shapes*, represented through $\pm\sqrt{\alpha}$, with associated fundamental modes y_1 (and y_2 when optimum λ is a double eigenvalue), are illustrated to suitable scale for selected values of $\bar{\alpha}$. The principal characteristics of these solutions are stated in the caption of Fig. 3, to which the reader is referred.

The fundamental modes y_1 and y_2 shown for the double optimum eigenvalue solutions in Fig. 3(c) and (d) are those determined along with the optimum area function $\alpha(x)$, eigenvalue λ and Lagrangian multipliers β and γ from the non-linear optimality eqns (15)–(20). Although the modes y_1 and y_2 belonging together are non-symmetrical and $y_1(x) \neq y_2(1-x)$, a symmetrical and an antisymmetrical linear combination of the modes can be constructed. This is due to the relationship (15) and the symmetry of α.

In Fig. 4, we indicate by curve ABCD the dimensionless buckling eigenvalue λ of the optimum solutions as a function of the minimum constraint $\bar{\alpha}$ in the interval $0 \leq \bar{\alpha} \leq 1$, where the end point values correspond to geometrically unconstrained and uniform column, respectively.

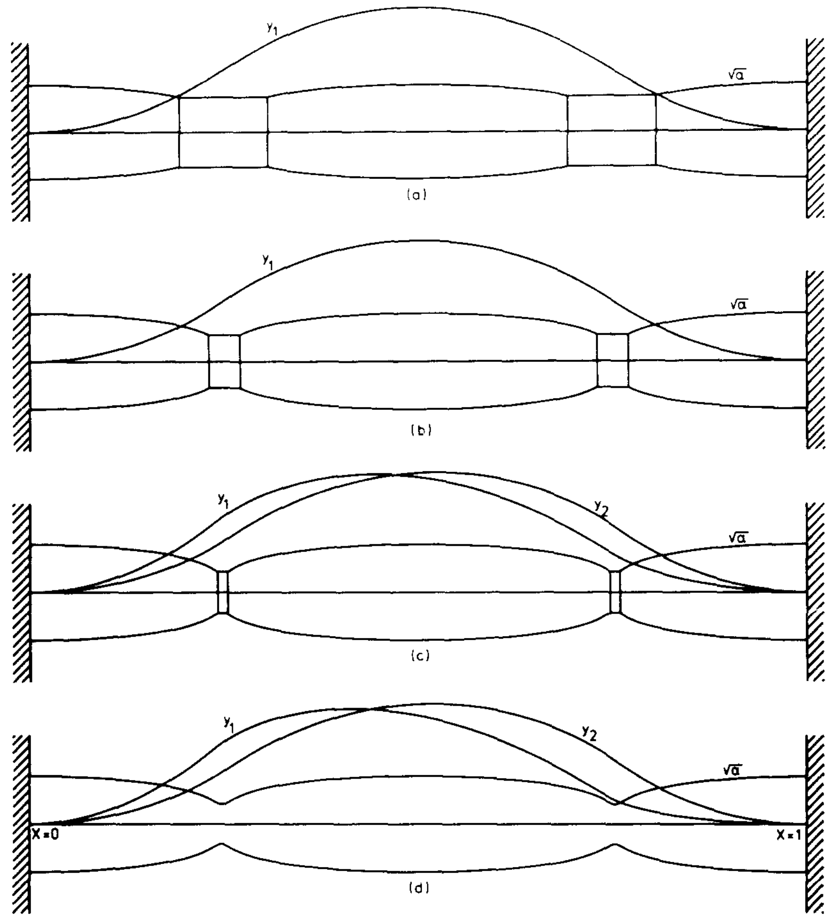

Fig. 3. Optimum column shapes and associated fundamental modes subject to different values of the geometric minimum constraint $\bar\alpha$: (a) $\bar\alpha = 0.7$. Optimum $\lambda = 48.690$ is simple, $\lambda/\lambda'' = 1.2333$. (b) $\bar\alpha = 0.4$. Optimum $\lambda = 51.775$ is simple, $\lambda/\lambda'' = 1.3115$. (c) $\bar\alpha = 0.25$. Optimum $\lambda = 52.349$ is double, $\lambda/\lambda'' = 1.3260$. The minimum constraint is still active. (d) The $\bar\alpha$ independent, optimum design for $0 \le \bar\alpha < 0.226$ (inactive minimum constraint). Optimum $\lambda = 52.3563$ is double, $\lambda/\lambda'' = 1.3262$. This is, in particular, the optimum solution for the geometrically unconstrained optimization problem.

Noting the abscissa $\bar\alpha = 0.280$ of point C, we find that the optimum buckling load is a simple eigenvalue for any value of $\bar\alpha$ belonging to the interval $0.280 < \bar\alpha \le 1$. The fundamental modes are symmetrical for these values of $\bar\alpha$. The corresponding optimum columns all consist of a middle part shaped like a constrained optimum simply supported column, viz. special cases in Refs. [3, 4], and two outer parts shaped like optimum clamped-free columns, each of which are identical with half the middle part. These three column parts, see e.g. Figs. 3(a) and (b), are mutually rigidly connected at the points $x = \frac{1}{4}$ and $x = \frac{3}{4}$, respectively, where the bending moment of the symmetrical mode vanishes, and where the geometric minimum constraint is active.

For the purpose of illustration, we have determined the second eigenvalues λ_2 of the optimum solutions subject to $0.280 < \bar\alpha \le 1$ by means of a common procedure of calculating eigenvalues and -modes. The results hereby obtained† are shown by curve CE in Fig. 4. For $\bar\alpha = 0.280$, the second eigenvalue λ_2 coalesces with the fundamental eigenvalue λ of the optimum column.

Now, for $\bar\alpha$ belonging to the interval $0 \le \bar\alpha \le 0.280$, we find that the optimum buckling load λ (curve ABC in Fig. 4) is associated with *two* modes y_1 and y_2, viz. Figs. 3(c) and (d).

For $0.226 \le \bar\alpha \le 1$ (see curve BCD in Fig. 4), optimum λ increases with decreasing geometric constraint. In addition, we find that the constraint $\bar\alpha$ is *active* in the corresponding design. However, for $\bar\alpha$ less than 0.226 (the abscissa of point B), this behaviour ceases.

†We also computed the eigenvalues corresponding to the third modes of the bimodal optimum solutions in order to check their assumed non-coalescense with the optimum eigenvalues.

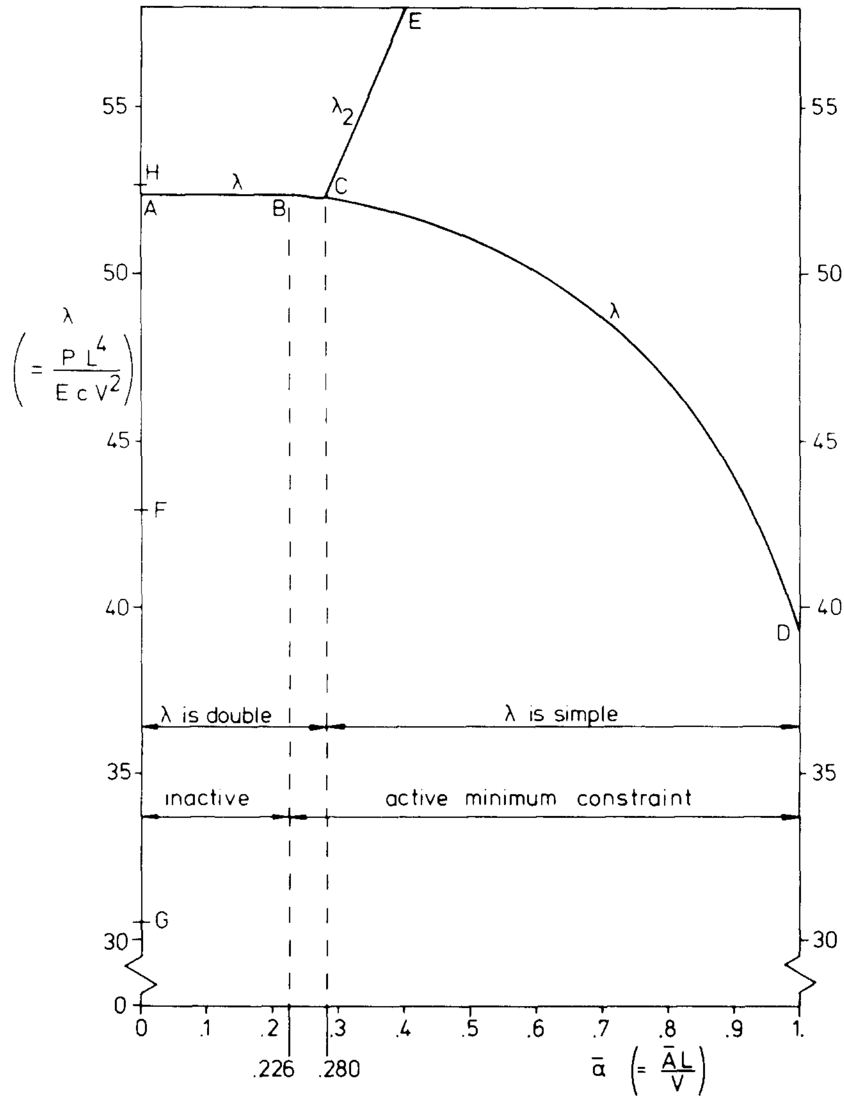

Fig. 4. Optimum buckling load $\lambda = PL^4/EcV^2$ (curve *ABCD*) vs minimum constraint $\bar{\alpha}$ for column cross-sectional area. Curve *CE* indicates second eigenvalues λ_2 of optimum columns for which λ is a simple eigenvalue.

Thus, for any value of $\bar{\alpha}$ in the interval $0 \leq \bar{\alpha} < 0.226$, see the horizontal line *AB* in Fig. 4, the constraint is *inactive*, and we obtain the *same* optimum design and associated (double) buckling load.

This $\bar{\alpha}$-independent bimodal optimum design, which is shown in Fig. 3(*d*), *constitutes our solution of the geometrically unconstrained optimization problem, and it replaces the solutions discussed in Section* 1. In contrast to the previous results, this geometrically unconstrained optimum column has finite cross-section throughout.

In a wider perspective, it is worth noting that our result provides an example of a *statically indeterminate* solution of a geometrically unconstrained, one-dimensional, single-purpose, structural optimization problem. Clearly, this *exception* from the usual class of statically determinate optimum solutions for such problems is closely connected with the form of the bimodal optimality condition, see e.g. eqn (15). This equation does not generally predict a vanishing cross-section with an inner hinge at interior points of zero bending moment for a geometrically unconstrained optimum column.

Now, the optimum (bimodal) buckling load of our geometrically unconstrained optimum column is found to be $\lambda = 52.3563$, which is 32.62% higher than the buckling load of a corresponding uniform column. In Fig. 4, the point *A* represents this result. For comparison, the points H and G represent the *overestimated* and the *actual* fundamental buckling load, respectively, of the *non-optimum* solution in Ref. [1], see Fig. 1. Moreover, point F represents the result shown in Fig. 2.

Concluding non-optimality of the columns in Figs. 1 and 2 in relation to the problem under study, we should mention that these columns may, *instead,* be interpreted as optimum solutions for slightly *different* optimization problems. Thus, it is easily shown that the column in Ref. [1] (Fig. 1) satisfies the necessary conditions of the problem of optimizing the *second* buckling eigenvalue of a clamped column equipped with two inner hinges. In Refs. [5, 6], a similar type of problem, namely the optimization of transversely vibrating beams (or rotating shafts) with respect to higher order natural frequencies (critical speeds), is dealt with. Similarly, the column in Fig. 2 can be interpreted as the solution to the problem of optimizing the fundamental buckling load of a clamped column with two inner hinges.

It is interesting to note that the second buckling eigenvalue of the column in Fig. 1 and the fundamental eigenvalue of the column in Fig. 2 are both bimodal, since the middle parts of these columns may, e.g. equally well deflect to the opposite side of the straight line connecting the hinges in their buckled positions. In fact, the two inner hinges of these columns make the bimodal optimality condition equivalent to the single mode optimality condition from which these columns have emerged.

We should finally remark that the present study clearly indicates that the *sufficiency proof* for optimality presented in Ref. [1] is based on too restrictive *a priori* assumptions for the clamped column case. In fact, Masur[7] and Popelar[8] have previously noted that the proof breaks down for statically indeterminate columns. It would be worthwhile to investigate whether a proof without loss of generality could be established.

REFERENCES

1. I. Tadjbakhsh and J. B. Keller, Strongest columns and isoperimetric inequalities for eigenvalues. *J. Appl. Mech.*, **29**, 159–164 (1962).
2. J. B. Keller, The shape of the strongest column. *Arch. Rational Mech. Anal.* **5**, 275–285 (1960).
3. J. C. Frauenthal, Constrained optimal design of columns against buckling. *J. Struct. Mech.*, **1**, 79–89 (1972).
4. S. H. Rasmussen, On the optimal shape of an elastic-plastic column. *J. Struct. Mech.* **4**, 307–320 (1976).
5. N. Olhoff, Optimization of vibrating beams with respect to higher order natural frequencies. *J. Struct. Mech.* **4**, 87–122 (1976).
6. N. Olhoff, Maximizing higher order eigenfrequencies of beams with constraints on the design geometry. *J. Struct. Mech.*, **5**(2), (1977) (To appear).
7. E. F. Masur, Optimal placement of available sections in structural eigenvalue problems. *J. Optimiz. Theory Applic.* **15**, 69–84 (1975).
8. C. H. Popelar, Optimal design of structures against buckling: a complementary energy approach. To appear in *J. Struct. Mech.* **5**(1), (1977).

APPENDIX

The method used for solving the governing eqns (15)–(20) of the bimodal eigenvalue optimization problem is outlined here. It consists of a numerical procedure of successive iterations based on a finite difference formulation of equations obtained by a formal integration of the problem. We first describe the formal integration and then present the iteration scheme used for successive iterations.

Integrating twice the buckling differential eqns (16) for the modes y_1 and y_2, we obtain

$$\alpha^2 y_i'' = \lambda(-y_i + a_i + b_i x) \qquad i = 1, 2. \tag{A1}$$

where a_i and b_i, $i = 1, 2$, are integration constants. Since the problem is statically indeterminate, these constants are to be determined later by kinematic boundary conditions.

Now, dividing the two equations comprised in (A1) by each other, we get the equation

$$y_2'' = \frac{-y_1 + a_1 + b_1 x}{-y_2 + a_2 + b_2 x} y_1'', \tag{A2}$$

which may be combined with eqns (A1) and eqn (15) so as to separate y_1'' and y_2'' (which, for $x \in x_u$, are coupled through α) and to express these derivatives in terms of y_i, a_i, b_i, λ, β and γ, respectively. Substituting these expressions for y_1'' and y_2'' into eqn (15), we then obtain the following convenient formula for $\alpha(x)$,

$$\alpha(x) = \begin{cases} \dfrac{\lambda^{2/3}}{\beta^{1/3}} \{(1-\gamma)(-y_1 + a_1 + b_1 x)^2 + \gamma(-y_2 + a_2 + b_2 x)^2\}^{1/3} \quad (\text{if} > \bar{\alpha}) & x \in x_u \\ \bar{\alpha} & x \in x_c \end{cases}. \tag{A3}$$

Thus, y_1'' and y_2'' are simply identified by

$$y_i''(x) = \lambda \frac{-y_i + a_i + b_i x}{\alpha^2} \quad i = 1, 2, \tag{A4}$$

see (A1), where $\alpha(x)$ is given by (A3). Imposing the boundary conditions at $x = 0$, see eqn (16), two integrations of (A4) yield

$$y_i'(x) = \lambda\left\{-\int_0^x y_i\alpha^{-2}\,d\xi + a_i\int_0^x \alpha^{-2}\,d\xi + b_i\int_0^x \xi\alpha^{-2}\,d\xi\right\} \quad i = 1, 2 \tag{A5}$$

$$y_i(x) = \lambda\left\{-\int_0^x\int_0^\xi y_i\alpha^{-2}\,d\eta\,d\xi + a_i\int_0^x\int_0^\xi \alpha^{-2}\,d\eta\,d\xi + b_i\int_0^x\int_0^\xi \eta\alpha^{-2}\,d\eta\,d\xi\right\} \quad i = 1, 2, \tag{A6}$$

where b_i and a_i are determined as follows by the boundary conditions at $x = 1$,

$$\left.\begin{aligned} b_i &= \frac{\int_0^1\int_0^\xi \alpha^{-2}\,d\eta\,d\xi \int_0^1 y_i\alpha^{-2}\,d\xi - \int_0^1\int_0^\xi y_i\alpha^{-2}\,d\eta\,d\xi\int_0^1 \alpha^{-2}\,d\xi}{\int_0^1\int_0^\xi \alpha^{-2}\,d\eta\,d\xi\int_0^1 \xi\alpha^{-2}\,d\xi - \int_0^1\int_0^\xi \eta\alpha^{-2}\,d\eta\,d\xi\int_0^1 \alpha^{-2}\,d\xi} \\ a_i &= \frac{\int_0^1 y_i\alpha^{-2}\,d\xi - b_i\int_0^1 \xi\alpha^{-2}\,d\xi}{\int_0^1 \alpha^{-2}\,d\xi} \end{aligned}\right\} \quad i = 1, 2. \tag{A7}$$

To solve the complete set of equations, i.e. (15)–(20), for the general bimodal eigenvalue optimization problem associated with a given value of $\bar{\alpha}$, we now use the following scheme for successive iterations:

START
- Take $y_1(x)$, $y_2(x)$, λ, β, γ, a_1, b_1, a_2 and b_2 arbitrarily.
- (I) Compute $\alpha(x)$ and thus x_u and x_c by (A3).
- (II) Compute a_1, b_1, a_2 and b_2 by (A7).
- (III) Compute $y_1''(x)$ and $y_2''(x)$ by (A4), $y_1'(x)$ and $y_2'(x)$ by (A5), and $y_1(x)$ and $y_2(x)$ by (A6) respectively. Normalize these functions such that (17) is satisfied.
- (IV) Determine γ from (19). (This is done by an inner loop of successive iterations).
- (V) Compute β by (18).
- (VI) Compute λ by (20).
- Go to (I) if the iterates have not converged.

END

The special case of single mode eigenvalue optimization may be specified by setting $y_2''(x) \equiv y_2'(x) \equiv y_2(x) \equiv 0$ throughout, setting $\gamma = 0$, and bypassing (IV).

The numerical solution procedure is based on a discrete representation of the iterates at a number of equally spaced stations in the interval $0 \le x \le 1$, and the sequence of iterates rapidly converges to the optimum solution.

In order to obtain accurate estimates of the locations of the constrained intervals and for precise modelling of functional behaviour, the number of stations was taken to be large, and the computations were performed in double precision. The numerical stability of the solutions obtained was checked through computations based on different numbers of stations.